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Solution of the relativistic Dirac–Hulthén problem

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Abstract

The one-particle three-dimensional Dirac equation with spherical symmetry is solved for the Hulthén potential. The s-wave relativistic energy spectrum and two-component spinor wavefunctions are obtained analytically. Conforming to the standard feature of the relativistic problem, the solution space splits into two distinct subspaces depending on the sign of a fundamental parameter in the problem. Unique and interesting properties of the energy spectrum are pointed out and illustrated graphically for several values of the physical parameters. The square integrable two-component wavefunctions are written in terms of the Jacobi polynomials. The nonrelativistic limit reproduces the well-known nonrelativistic energy spectrum and results in Schrödinger equation with a 'generalized' three-parameter Hulthén potential, which is the sum of the original Hulthén potential and its square.

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1. Introduction

The Hulthén potential [1] is of considerable significance to various applications in many areas of physics. This includes applications in nuclear and particle physics, atomic physics, condensed matter and chemical physics. One may consult, for example, the literature cited in the papers of [2]. For spherically symmetric interaction, this potential has the general form $(1 - e^{\omega r})^{-1}$, where ω is a real positive parameter. A well-known application of this potential in atomic and nuclear physics is the screened Coulomb potential. In this case it is written as $V(r) = -\frac{\omega Z e^{-\omega r}}{1-e^{-\omega r}}$, where Z is the Coulomb charge and ω is the screening parameter. For small values of the screening parameter it is approximated as $V(r) \approx -\frac{Z}{r} e^{-\omega r}$, showing clearly the screening effect. The nonrelativistic Hulthén problem has a closed-form analytic solution only for s-wave ($\ell = 0$) [3]. Several techniques were used to obtain approximate solutions in the case where the angular momentum is not zero [2, 4, 5]. Various methods were employed in obtaining the solution (energy spectrum and wavefunction) of the nonrelativistic

Hulthén problem. Super-symmetric quantum mechanics, shape invariance, path integration and dynamical group are four methods among many which were used in the search for exact solutions of the wave equation with the Hulthén potential.

The relativistic problem, on the other hand, did not receive adequate attention. Most of the limited work on this problem was in the context of the Klein–Gordon equation. The Klein–Gordon equation with the Hulthén potential was treated by Znojil [6]. The same problem but with both vector and scalar Hulthén-type potentials was later investigated more thoroughly by Domínguez–Adame [7]. The Green function for the Klein–Gordon operator with these two potentials was obtained using path-integral approach by Chetouani *et al* [8]. The scattering state solutions of the s-wave Klein–Gordon equation with vector and scalar Hulthén potentials were obtained for regular and irregular boundary conditions by Talukdar *et al* [9]. On the other hand, only very few articles were written on the Dirac equation with the Hulthén potential. Roy and Roychoudhury used the algebraic approach in tackling this problem [10]. They considered the Dirac equation with scalar and vector potentials with the space component of the later being zero. Both potentials are of the same Hulthén type but of different strengths. This made the potential in the Dirac equation look like $\left(\begin{smallmatrix} V+S & 0 \\ 0 & V-S \end{smallmatrix}\right)$, where V is the time component of the vector potential and S is the scalar. Explicit results were obtained for $S = V$ making the potential in the Dirac equation of the form $\left(\begin{smallmatrix} 2V & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and resulting in Schrödinger-like equation for the upper spinor component. However, it is believed that such a potential structure might result in a singular behaviour of the solution. A remedy could be found in the introduction of an additional pseudo-scalar potential W such that the overall potential structure in the Dirac equation becomes $\left(\begin{smallmatrix} V+S & W \\ W & V-S \end{smallmatrix}\right)$. Thus, taking $S = V$ results in the more regular potential form $\left(\begin{smallmatrix} 2V & W \\ W & 0 \end{smallmatrix}\right)$ while at the same time giving the sought after Schrödinger-like equation. Quite recently, Guo *et al* presented a more appropriate treatment of the relativistic s-wave problem in the Dirac equation with non-minimal coupling to spherically symmetric vector potential of the Hulthén type [11]. They obtained the bound-states energy spectrum and spinor wavefunctions in terms of the hypergeometric function.

In this paper, we present a systematic development of the solution of this relativistic problem in the context of the Dirac equation using the same approach followed by Guo *et al*. However, we complement their solution by giving a complete one where the solution space splits into two distinct subspaces, one of which was obtained by the authors but not the other. The splitting depends on the sign of a fundamental parameter in the problem, which is a standard feature of the relativistic problem. For example, it is well known that in relativistic problems with nonzero angular momentum two regular solutions are obtained depending on the sign of κ, where $\kappa = \pm 1, \pm 2, \ldots$ for $\ell = j \pm \frac{1}{2}$. However, the current problem is analytically solvable only for $\ell = 0$. Nonetheless, we do find that in this case as well the solution space splits depending, however, on the sign of another dimensionless physical parameter symbolized by ζ. The solution of the problem which was obtained in [11] is only for $\zeta < 0$ (in [11], the parameter λ corresponds to $-\zeta$). Furthermore, the partial analysis of the energy spectrum which was given there for $\zeta = -1$ will be extended here to all values of ζ. Interesting and unique properties of the spectrum will be pointed out and graphical representations will be given for several values of the physical parameters. The square integrable two-component spinor wavefunctions are written in terms of the Jacobi polynomials for all ζ. Taking the nonrelativistic limit reproduces the nonrelativistic spectrum and gives an s-wave Schrödinger equation with a 'generalized' three-parameter Hulthén potential, which is the sum of the original Hulthén potential and its square.

The paper is organized as follows. In section 2, we set up the three-dimensional Dirac equation for a spinor coupled in a non-minimal way to the four-potential $(\mathcal{A}_0, \vec{\mathcal{A}})$. Spherical symmetry is imposed, which reduces the problem to a solution of the 2×2 radial Dirac equation

involving two real radial functions, one of which is the independent potential function of the problem. A global unitary transformation is applied to the radial Dirac equation to separate the variables such that the resulting second-order differential equation for the spinor components becomes Schrödinger-like. This requirement results in a linear constraint which relates the auxiliary radial function to the independent potential function. In section 3 this scheme is implemented in the case where the potential function, which is the time component of the 4-vector $(\mathcal{A}_0, \vec{\mathcal{A}})$, is taken as the Hulthén potential. The resulting s-wave Schrödinger-like equation is solved by comparison with that for an $L^2[-1,+1]$ function which is written in terms of the Jacobi polynomial. The solution of the problem (the upper and lower components of the spinor wavefunction and the relativistic bound states energy spectrum) is obtained for all values of the parameter ζ. The paper concludes with a brief analysis of the energy spectrum.

2. Solution approach for the Dirac equation

In the atomic units $\hbar = m = 1$, the three-dimensional Dirac Hamiltonian for a four-component spinor with 'minimal' coupling to the time-independent vector potential $\mathcal{A}_\mu = (\mathcal{A}_0, \vec{\mathcal{A}})$ reads

$$H = \begin{pmatrix} \lambda\!\!\!^{-2}\mathcal{A}_0 + 1 & -\mathrm{i}\lambda\!\!\!^{-}\vec{\sigma}\cdot\vec{\nabla} + \lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}} \\ -\mathrm{i}\lambda\!\!\!^{-}\vec{\sigma}\cdot\vec{\nabla} + \lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}} & \lambda\!\!\!^{-2}\mathcal{A}_0 - 1 \end{pmatrix} \tag{2.1}$$

where $\lambda\!\!\!^{-}$ is the Compton wavelength $\hbar/mc = c^{-1}$ and $\vec{\sigma}$ are the three 2×2 Hermitian Pauli matrices. H is measured in units of the rest mass, mc^2. Gauge invariance could be used to eliminate the contribution of the off-diagonal term $\lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}}$ in the Hamiltonian (2.1). However, our choice of coupling will be non-minimal, which is accomplished by the replacement $\lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}} \to \pm\mathrm{i}\lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}}$, respectively. That is the Hamiltonian (2.1) is replaced by the following:

$$H = \begin{pmatrix} \lambda\!\!\!^{-2}\mathcal{A}_0 + 1 & -\mathrm{i}\lambda\!\!\!^{-}\vec{\sigma}\cdot\vec{\nabla} + \mathrm{i}\lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}} \\ -\mathrm{i}\lambda\!\!\!^{-}\vec{\sigma}\cdot\vec{\nabla} - \mathrm{i}\lambda\!\!\!^{-2}\vec{\sigma}\cdot\vec{\mathcal{A}} & \lambda\!\!\!^{-2}\mathcal{A}_0 - 1 \end{pmatrix}. \tag{2.2}$$

It should be noted that this type of coupling does not support an interpretation of $(\mathcal{A}_0, \vec{\mathcal{A}})$ as the electromagnetic potential unless, of course, $\vec{\mathcal{A}} = 0$ (e.g., the Coulomb potential). Likewise, H will not have local gauge symmetry. That is, the associated wave equation is not invariant under the electromagnetic gauge transformation.

We impose spherical symmetry and write $(\mathcal{A}_0, \vec{\mathcal{A}})$ as $\left[V(r), \frac{1}{\lambda\!\!\!^{-}}\hat{r}\,W(r)\right]$, where $\hat{r}$ is the radial unit vector. $V(r)$ and $W(r)$ are real radial functions referred to as the even and odd components of the relativistic potential, respectively. Because of the spherical symmetry the angular variables could be separated and we can write the radial Dirac equation as

$$\begin{pmatrix} +1 + \lambda\!\!\!^{-2}V(r) - \varepsilon & \lambda\!\!\!^{-}\left[\frac{\kappa}{r} + W(r) - \frac{\mathrm{d}}{\mathrm{d}r}\right] \\ \lambda\!\!\!^{-}\left[\frac{\kappa}{r} + W(r) + \frac{\mathrm{d}}{\mathrm{d}r}\right] & -1 + \lambda\!\!\!^{-2}V(r) - \varepsilon \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0 \tag{2.3}$$

where κ is the spin–orbit quantum number defined as $\kappa = \pm\left(j + \frac{1}{2}\right) = \pm 1, \pm 2, \ldots$ for $\ell = j \pm \frac{1}{2}$ and ε is the relativistic energy which is measured in units of mc^2. f and g are real radial square-integrable functions. This matrix equation results in two coupled first-order differential equations for the two radial spinor components. Eliminating one component in favour of the other gives a second-order differential equation. This will not be Schrödinger-like (i.e. it contains first-order derivatives) unless $V = 0$. To obtain a Schrödinger-like equation in the general case we proceed as follows. A global unitary transformation $\mathcal{U}(\eta) = \exp\left(\frac{\mathrm{i}}{2}\lambda\!\!\!^{-}\eta\sigma_2\right)$ is applied to the Dirac equation (2.3), where η is a real constant parameter and σ_2 is the 2×2 Pauli matrix $\left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$. The Schrödinger-like requirement relates the two potential components by the linear constraint $V(r) = \xi[W(r) + \kappa/r]$, where ξ is a real parameter. It also requires that $\sin(\lambda\!\!\!^{-}\eta) = \pm\lambda\!\!\!^{-}\xi$, where $-\frac{\pi}{2} < \lambda\!\!\!^{-}\eta < +\frac{\pi}{2}$. This results in a Hamiltonian that will be written in

terms of only one arbitrary potential function, either the even potential component $V(r)$ or the odd one $W(r)$. Moreover, the solution of the problem is obtained for a given value of κ.

The unitary transformation together with the potential constraint maps equation (2.3) into the following one, which we choose to write in terms of the even potential component:

$$\begin{pmatrix} C-\varepsilon+(1\pm 1)\lambda^2 V & \lambda\left(\mp\xi+\frac{C}{\xi}V-\frac{\mathrm{d}}{\mathrm{d}r}\right) \\ \lambda\left(\mp\xi+\frac{C}{\xi}V+\frac{\mathrm{d}}{\mathrm{d}r}\right) & -C-\varepsilon+(1\mp 1)\lambda^2 V \end{pmatrix}\begin{pmatrix}\phi^+(r)\\ \phi^-(r)\end{pmatrix}=0 \tag{2.4}$$

where $C=\cos(\lambda\eta)=\sqrt{1-(\lambda\xi)^2}>0$ and

$$\begin{pmatrix}\phi^+\\ \phi^-\end{pmatrix}=\mathcal{U}\psi=\begin{pmatrix}\cos\frac{\lambda\eta}{2} & \sin\frac{\lambda\eta}{2}\\ -\sin\frac{\lambda\eta}{2} & \cos\frac{\lambda\eta}{2}\end{pmatrix}\begin{pmatrix}g\\ f\end{pmatrix}. \tag{2.5}$$

Equation (2.4) gives the following equation for one spinor component in terms of the other

$$\phi^{\mp}(r)=\frac{\lambda}{C\pm\varepsilon}\left[-\xi\pm\frac{C}{\xi}V(r)+\frac{\mathrm{d}}{\mathrm{d}r}\right]\phi^{\pm}(r). \tag{2.6}$$

While, the resulting Schrödinger-like wave equation becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left(\frac{C}{\xi}\right)^2V^2\mp\frac{C}{\xi}\frac{\mathrm{d}V}{\mathrm{d}r}+2\varepsilon V-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^{\pm}(r)=0. \tag{2.7}$$

The top and bottom signs in the above equations correspond to the choice of sign in $\sin(\lambda\eta)=\pm\lambda\xi$. In the nonrelativistic limit ($\lambda\to 0$), $\varepsilon\approx 1+\lambda^2E$ and $C\approx 1-\frac{1}{2}\lambda^2\xi^2$. Therefore, equation (2.6) shows that ϕ^+ is the larger of the two relativistic spinor components (i.e. ϕ^+ is the component that survives the nonrelativistic limit, whereas $\phi^-\sim\lambda\phi^+\to 0$). Consequently, if we favour the upper spinor component then our choice of sign in the transformation parameter constraint is the top (+) sign. That is, we choose $\sin(\lambda\eta)=+\lambda\xi$.

In all relativistic problems successfully tackled so far by this approach [12], equation (2.7) is solved by correspondence with well-known exactly solvable nonrelativistic problems. This correspondence results in a parameter map that relates the two problems. Now, if the nonrelativistic problem is exactly solvable, then using this parameter map and the known nonrelativistic energy spectrum, one can easily obtain the relativistic spectrum. In fact, the relativistic extension of any known dynamical relation-ship in the nonrelativistic theory could easily be obtained by this correspondence map. The Green function, which has prime significance in the calculation of relativistic processes, is such an example [13]. Moreover, the spinor wavefunction is also obtained from the nonrelativistic wavefunction using the same parameter map. In the following section this approach will be used in finding the solution of the Dirac equation with $V(r)$ taken as the Hulthén potential.

An alternative, but equivalent, approach to the one given above is to postulate the one-parameter two-component equation (2.4) as the relativistic wave equation and show that in the nonrelativistic limit ($\lambda\to 0$) the nonrelativistic problem is recovered. However, in this case, one cannot claim that the relativistic problem is a unique extension of the nonrelativistic one.

3. The Dirac–Hulthén problem

The radial equation for the Dirac–Hulthén problem is equation (2.4) with the upper signs and with $V(r)=-A/(\mathrm{e}^{\omega r}-1)$, where ω is the effective screening range of the potential. A and ω are real and ω positive. This problem was partially solved for s-wave ($\ell=0$) by Guo *et al* using the same approach presented above [11]. Equation (2.7) for the upper spinor component gives

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\rho(\rho+\omega)}{(\mathrm{e}^{\omega r}-1)^2}+\frac{\rho\omega-2\varepsilon A}{\mathrm{e}^{\omega r}-1}-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^+(r)=0 \tag{3.1}$$

where $\rho = \tau\sqrt{1-(\lambda A/\tau)^2}$ and $\tau = -A/\xi$. If we choose the transformation parameter ξ such that $\rho = -\omega$ then we obtain the solution for a pure s-wave Dirac–Hulthén problem. Nevertheless, we consider here the solution of a 'generalized' three-parameter Dirac–Hulthén problem where ρ is arbitrary and the potential consists of the sum of two terms: the original Hulthén potential and its square. Taking $x = 1 - 2\mathrm{e}^{-\omega r}$ maps real space into a bounded one. That is, $r \in [0, \infty] \to x \in [-1, +1]$. A square integrable function (with respect to the measure $\mathrm{d}r = \frac{1}{\omega}\frac{\mathrm{d}x}{1-x}$) in this configuration space that is compatible with the domain of the wave operator and satisfies the boundary conditions could be written as

$$\phi_n^+(r) = a_n(1+x)^\alpha(1-x)^\beta P_n^{(\mu,\nu)}(x) \tag{3.2}$$

where $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial and a_n is the normalization constant

$$a_n = \sqrt{\frac{\omega(2n+\mu+\nu+1)}{2^{\mu+\nu+1}}\frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}}. \tag{3.3}$$

The real parameters $\alpha, \beta > 0$ and $\mu, \nu > -1$. Using the differential equation, differential formulae of the Jacobi polynomial [14] and $\frac{\mathrm{d}}{\mathrm{d}r} = \omega(1-x)\frac{\mathrm{d}}{\mathrm{d}x}$ we can write

$$\begin{aligned}\frac{\mathrm{d}^2\phi_n^+}{\mathrm{d}r^2} = \omega^2\frac{1-x}{1+x}\Bigg\{\Bigg[&-n\left(x+\frac{\nu-\mu}{2n+\mu+\nu}\right)\left(\frac{\mu-2\beta}{1-x}+\frac{2\alpha-\nu-1}{1+x}\right)\\ &-n(n+\mu+\nu+1)-\alpha(2\beta+1)+\beta^2\frac{1+x}{1-x}+\alpha(\alpha-1)\frac{1-x}{1+x}\Bigg]\phi_n^+\\ &+2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\beta}{1-x}+\frac{2\alpha-\nu-1}{1+x}\right)\frac{a_n}{a_{n-1}}\phi_{n-1}^+\Bigg\}.\end{aligned} \tag{3.4}$$

To eliminate the off-diagonal representation (the ϕ_{n-1}^+ term) we should take $\mu = 2\beta$ and $\nu = 2\alpha - 1$. Consequently, we obtain the following second-order differential equation:

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\omega^2\alpha(\alpha-1)\left(\frac{1-x}{1+x}\right)^2-\omega^2[n(n+2\alpha+\mu)+\alpha(\mu+1)]\frac{1-x}{1+x}+\frac{\mu^2\omega^2}{4}\right\}\phi_n^+ = 0. \tag{3.5}$$

Comparing this with equation (3.1) results in the following parameter assignment:

$$\alpha = \begin{cases} 1+\zeta & \tau \geqslant 0 \\ -\zeta & \tau < 0 \end{cases} \qquad \mu_n = \begin{cases} \lambda_{n+1}^\zeta & \tau \geqslant 0 \\ \lambda_n^{-\zeta} & \tau < 0 \end{cases} \tag{3.6}$$

where $\zeta \equiv \rho/\omega$ and $\lambda_n^\zeta = -(n+\zeta) + (n+\zeta)^{-1}(\zeta^2 + 2\varepsilon A/\omega^2)$. Moreover, the comparison of the two equations gives, as well, the following quadratic (in ε) parameter relation:

$$\varepsilon^2 = 1 - (\lambda\omega/2)^2\mu_n^2 \qquad n = 0, 1, 2, \ldots, n_{\max}. \tag{3.7}$$

The energy spectrum is obtained as the set of two real solutions $\left\{\varepsilon_n^\pm\right\}_{n=0}^{n_{\max}}$ of this quadratic equation, where $n_{\max}$ is the maximum integer n that yields real solutions for equation (3.7).

The upper radial spinor component is obtained by substituting the parameters of (3.6) into the wavefunction (3.2) giving

$$\phi_n^+(r) = \begin{cases} a_n^{\zeta+1}\sqrt{2^{\lambda_{n+1}^\zeta}}\,\mathrm{e}^{-\omega\lambda_{n+1}^\zeta r/2}(1-\mathrm{e}^{-\omega r})^{\zeta+1}P_n^{(\lambda_{n+1}^\zeta, 2\zeta+1)}(1-2\mathrm{e}^{-\omega r}) & \tau \geqslant 0 \\ a_n^{-\zeta}\sqrt{2^{\lambda_n^{-\zeta}}}\,\mathrm{e}^{-\omega\lambda_n^{-\zeta} r/2}(1-\mathrm{e}^{-\omega r})^{-\zeta}P_n^{(\lambda_n^{-\zeta}, -2\zeta-1)}(1-2\mathrm{e}^{-\omega r}) & \tau < 0. \end{cases} \tag{3.8}$$

The lower component of the spinor wavefunction is obtained by substituting this in the 'kinetic balance' relation (2.6) which, in the x-coordinate, reads as follows,

$$\phi^-(r) = \frac{(\lambda\omega)}{\varepsilon+\rho/\tau}\left[\frac{A}{\tau\omega}+\zeta\left(\frac{1-x}{1+x}\right)+(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\right]\phi^+(r) \tag{3.9}$$

for $\varepsilon \neq -\rho/\tau$. Using the following recursion relations and differential formula satisfied by the Jacobi polynomials [14],

$$\left(\frac{1+x}{2}\right) P_n^{(\mu,\nu)} = \frac{n+\nu}{2n+\mu+\nu+1} P_n^{(\mu,\nu-1)} + \frac{n+1}{2n+\mu+\nu+1} P_{n+1}^{(\mu,\nu-1)} \tag{3.10a}$$

$$P_n^{(\mu,\nu)} = \frac{n+\mu+\nu+1}{2n+\mu+\nu+1} P_n^{(\mu,\nu+1)} + \frac{n+\mu}{2n+\mu+\nu+1} P_{n-1}^{(\mu,\nu+1)} \tag{3.10b}$$

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\mu,\nu)} = -n\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right) P_n^{(\mu,\nu)} + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu} P_{n-1}^{(\mu,\nu)} \tag{3.10c}$$

and writing the energy eigenvalues as

$$\varepsilon_n = \begin{cases} \varepsilon_{n+1}^{\zeta} & \tau \geqslant 0 \\ \varepsilon_n^{-\zeta} & \tau < 0 \end{cases}$$

we obtain the following expressions for the lower spinor component when $\tau \geqslant 0$:

$$\begin{aligned} \phi_n^-(r) = \frac{\lambda\!\!\!^{-}\omega a_n^{\zeta+1}}{\varepsilon_{n+1}^{\zeta} + \rho/\tau} \sqrt{2^{\lambda_{n+1}^{\zeta}}}\, \mathrm{e}^{-\omega\lambda_{n+1}^{\zeta} r/2} (1-\mathrm{e}^{-\omega r})^{\zeta} \Bigg[\left(\frac{-\zeta + A/\tau\omega}{n+1+\zeta+\lambda_{n+1}^{\zeta}/2} + 1 \right) \\ \times (n+2\zeta+1) P_n^{(\lambda_{n+1}^{\zeta}, 2\zeta)}(1-2\mathrm{e}^{-\omega r}) + \left(\frac{-\zeta + A/\tau\omega}{n+1+\zeta+\lambda_{n+1}^{\zeta}/2} - 1 \right) \\ \times (n+1) P_{n+1}^{(\lambda_{n+1}^{\zeta}, 2\zeta)}(1-2\mathrm{e}^{-\omega r}) \Bigg]. \end{aligned} \tag{3.11a}$$

For $\tau < 0$, the result is

$$\begin{aligned} \phi_n^-(r) = -\frac{2\lambda\!\!\!^{-}\omega a_n^{-\zeta}}{\varepsilon_n^{-\zeta} + \rho/\tau} \sqrt{2^{\lambda_n^{-\zeta}}}\, \mathrm{e}^{-\omega\lambda_n^{-\zeta} r/2} (1-\mathrm{e}^{-\omega r})^{-\zeta} \Big\{ \big(n + \lambda_n^{-\zeta}/2 - A/\tau\omega\big) \\ \times P_n^{(\lambda_n^{-\zeta}, -2\zeta-1)}(1-2\mathrm{e}^{-\omega r}) - (1-\mathrm{e}^{-\omega r})^{-1} \big[(2\zeta+1) P_n^{(\lambda_n^{-\zeta}, -2\zeta-1)}(1-2\mathrm{e}^{-\omega r}) \\ + (n-2\zeta-1) P_{n+1}^{(\lambda_n^{-\zeta}, -2\zeta-2)}(1-2\mathrm{e}^{-\omega r}) \big] \Big\}. \end{aligned} \tag{3.11b}$$

Using the Jacobi polynomial identity $P_n^{(\mu,\nu)}(-1) = (-)^n \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\Gamma(\nu+1)}$ one can verify that the term inside the square brackets in (3.11*b*) vanishes at $r = 0$. Moreover, one can also show that

$$\lim_{\delta\to 0^+} \big[\nu P_n^{(\mu,\nu)}(-1+\delta) - (n+\nu) P_n^{(\mu,\nu-1)}(-1+\delta) \big] = \frac{(-)^n \Gamma(n+\nu+1)}{2\Gamma(n)\Gamma(\nu+2)} (n+\mu)\delta + O(\delta^2). \tag{3.12}$$

Therefore, the term with the $(1-\mathrm{e}^{-\omega r})^{-1}$ factor in (3.11*b*) is finite at $r = 0$ maintaining square integrability of $\phi_n^-(r)$.

We conclude with a brief analysis of the relativistic energy spectrum $\{\varepsilon_n^{\pm}\}_{n=0}^{n_{\max}}$. For the sake of clarity, this will be given in a list as follows:

(1) It can easily be shown that the two solutions of the quadratic equation (3.7) are

$$\begin{aligned} \varepsilon_n^{\pm} = \left[1 + \left(\frac{\lambda\!\!\!^{-} A/\omega}{n+\alpha} \right)^2 \right]^{-1} \Bigg\{ \frac{\lambda\!\!\!^{-2} A}{2} \left[1 - \left(\frac{\zeta}{n+\alpha} \right)^2 \right] \\ \pm \sqrt{ 1 + \left(\frac{\lambda\!\!\!^{-} A/\omega}{n+\alpha} \right)^2 - \left(\frac{\lambda\!\!\!^{-}\omega}{2} \right)^2 (n+\alpha)^2 \left[1 - \left(\frac{\zeta}{n+\alpha} \right)^2 \right]^2 } \Bigg\}. \end{aligned} \tag{3.13}$$

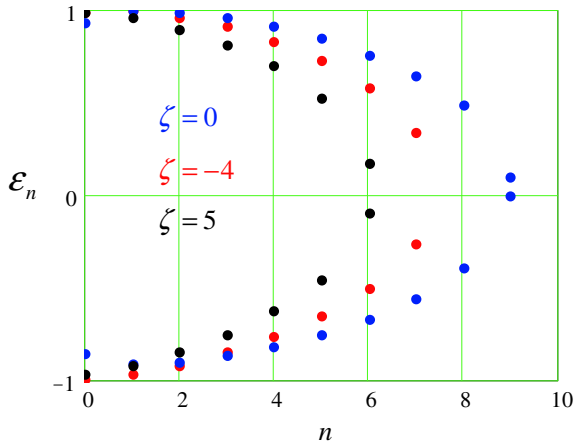

Figure 1. The relativistic energy spectrum of the s-wave 'generalized' Dirac–Hulthén problem for several values of the dimensionless parameter ζ. The other physical parameters are taken (in arbitrary length units for ω) as $A = 2.5\omega^2$ and $\lambda\!\!\!^{-} = 0.2\omega^{-1}$. The figure also shows that in this example $n_{\max} = 6, 7, 9$ for $\zeta = 5, -4, 0$, respectively.

(2) The reality constraint on this energy spectrum requires that the expression under the square root be positive. This results in the condition that $n \leqslant n_{\max}$, where $n_{\max}$ is the maximum integer that is less than or equal to

$$\left\{\zeta^2 + \frac{2}{(\lambda\!\!\!^{-}\omega)^2}\left[1 + \sqrt{1 + (\lambda\!\!\!^{-}\tau)^2}\right]\right\}^{1/2} - \alpha \tag{3.14}$$

where $\tau^2 = \zeta^2\omega^2 + \lambda\!\!\!^{-2}A^2$.

(3) The condition for the existence of bound states could then be translated into the condition that $n_{\max} \geqslant 0$.

(4) It is evident from (3.14) that $n_{\max} \to \infty$ in the nonrelativistic limit ($\lambda\!\!\!^{-} \to 0$) or in the no-screening limit ($\omega \to 0$).

(5) The energy spectrum is made up of two halves: the upper half which is the collection of all energy eigenvalues in the set $\{\varepsilon_n^+\}_{n=0}^{n_{\max}}$, and the lower half which is made up of the set $\{\varepsilon_n^-\}_{n=0}^{n_{\max}}$.

(6) The lower bound of the upper half of the spectrum will either be $\varepsilon_{n_{\max}}^+$ or ε_0^+. On the other hand, $\varepsilon_{n_{\max}}^-$ or ε_0^- will be the upper bound for the lower half. The choice of one or the other depends on the values of the physical parameters in the problem.

(7) Generally, for a symmetric spectrum, the two bounds match. That is, they go together either as $\varepsilon_{n_{\max}}^{\pm}$ or as $\varepsilon_0^{\pm}$.

(8) For a spectrum which is highly symmetric around the zero energy line, the two energy bounds are $\varepsilon_{n_{\max}}^{\pm}$.

(9) It is easy to verify that for $n = 0$ and when $\tau < 0$ equation (3.13) gives $\varepsilon_0^{\pm} = \pm\rho/\tau$.

(10) The four energy bounds $\{\varepsilon_0^{\pm}|_{\tau<0}, \varepsilon_{n_{\max}}^{\pm}|_{\tau\geqslant 0}\}$ correspond to non-degenerate states, while all others do not. This is so because equation (3.13) or, equivalently, equation (3.7) states that $\varepsilon_n^{\pm}|_{\tau\geqslant 0} = \varepsilon_{n+1}^{\pm}|_{\tau<0}$ for $n = 0, 1, 2, \ldots, n_{\max} - 1$ and for all τ.

Figure 1 shows the relativistic energy spectrum for a given set of physical parameters and for several values of the dimensionless parameter ζ. For large values of the potential strength A,

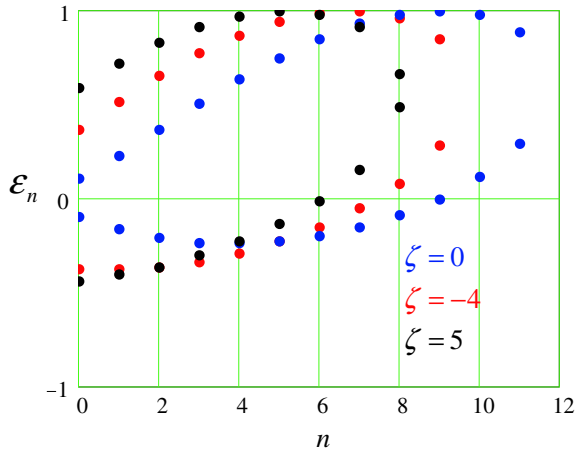

Figure 2. The energy spectrum for the same system as in figure 1 except for a substantial increase in the strength of the potential, where $A = 50\omega^2$. It is evident that the spectrum becomes less symmetric bending upward. Moreover, for this case $n_{\max} = 8, 9, 11$, respectively.

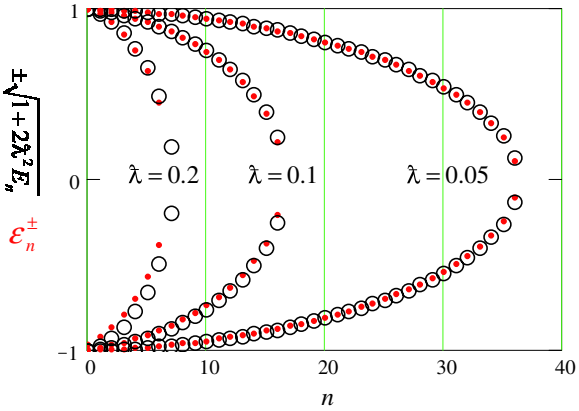

Figure 3. Shows that the relativistic energy spectrum (red dots) approaches the nonrelativistic spectrum (black circles) as λ becomes smaller and smaller (shown in units of $1/\omega$). The other parameters in this illustration were taken as $A = 2\omega^2$ and $\zeta = 3$.

the energy spectrum bends upward becoming less symmetric as illustrated in figure 2. By studying equation (3.7), one can easily show that in the nonrelativistic limit ($\lambda \to 0$) the following energy spectrum is obtained,

$$E_n = -\frac{\omega^2}{8}\left(n + \alpha - \frac{\zeta^2 + 2A/\omega^2}{n+\alpha}\right)^2 \qquad n = 0, 1, 2, \ldots \tag{3.15}$$

which agrees with the nonrelativistic results obtained in [5]. Figure 3 shows that in the nonrelativistic limit $\varepsilon_n \to E_n$. Specifically, using $\varepsilon_n^{\pm}$ from formula (3.13) and E_n from (3.15), we find that the smaller λ the better the following approximation:

$$\varepsilon_n^{\pm} \approx \pm\sqrt{1 + 2\lambda^2 E_n} \approx \pm(1 + \lambda^2 E_n). \tag{3.16}$$

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